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A MOMENT APPROACH TO ANALYZE ZEROS OF TRIANGULAR POLYNOMIAL SETS

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Abstract. Let $I = \langle g_1, \ldots, g_n \rangle$ be a zero-dimensional ideal of $\mathbb{R}[x_1, \ldots, x_n]$ such that its associated set $\mathbb{G}$ of polynomial equations $g_i(x) = 0$ for all $i = 1, \ldots, n$ is in triangular form. By introducing multivariate Newton sums we provide a numerical characterization of polynomials in $\sqrt{I}$. We also provide a necessary and sufficient (numerical) condition for all the zeros of $\mathbb{G}$ to be in a given set $\mathbb{K} \subset \mathbb{C}^n$, without explicitly computing the zeros. In addition, we also provide a necessary and sufficient condition on the coefficients of the g_i's for $\mathbb{G}$ to have (a) only real zeros, (b) to have only real zeros, all contained in a given semi-algebraic set $\mathbb{K} \subset \mathbb{R}^n$. In the proof technique, we use a deep result of Curto and Fialkow (2000) on the $\mathbb{K}$-moment problem, and the conditions we provide are given in terms of positive definiteness of some related moment and localizing matrices depending on the g_i's via the Newton sums of $\mathbb{G}$. In addition, the number of distinct real zeros is shown to be the maximal rank of a related moment matrix.

1. Introduction

In this paper we consider an ideal $I := \langle g_1, \ldots, g_n \rangle \subset \mathbb{R}[x_1, \ldots x_n]$ generated by the real-valued polynomials $g_i \in \mathbb{R}[x_1, \ldots, x_n]$. Let us call $\mathbb{G} := \{g_1, \ldots, g_n\}$ a *polynomial set* and let a term ordering of monomials with $x_1 < x_2 < \cdots < x_n$ be given.

We assume that the sytem of polynomials equations $\{g_i(x) = 0,\ i = 1, \ldots, n\}$ is in the following triangular form:

$$g_i(x) = p_i(x_1, \ldots, x_{i-1}) x_i^{r_i} + h_i(x_1, \ldots, x_i), \quad i = 1, \ldots, n, \tag{1.1}$$

by which we mean the following:

(i) x_i is the main variable and $p_i(x_1, \ldots, x_{i-1}) x_i^{r_i}$ is the leading term of g_i.

(ii) for every $i = 2, \ldots, n$, every zero in $\mathbb{C}^n$ of the polynomial system $\mathbb{G}_{i-1} := \{g_1, \ldots g_{i-1}\}$ is *not* a zero of the leading coefficient $\mathrm{ini}(g_i) := p_i(x_1, \ldots, x_{i-1})$ of g_i.

The set $\mathbb{G}$ is called a *triangular set*. From (i)-(ii), it follows that I is a zero-dimensional ideal. Conversely, any zero-dimensional ideal can be represented by a finite union of specific triangular sets (see e.g. Aubry et al. [1], Lazard [10]). For various definitions (and results) related to *triangular sets* (e.g. due to Kalkbrener, Lazard, Wu) the interested reader is referred to Lazard [10], Wang [7] and the many references therein; see also Aubry and Maza [2] for a comparison of symbolic algorithms related to triangular sets.

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For instance, there are symbolic algorithms that, given I as input, generate a finite set of triangular systems in the specific form $g_i(x) = x_i - f_i(x_1)$ for all $i = 2, \dots, n$. Triangular sets in the latter form are particularly interesting to develop efficient symbolic algorithms for counting and computing real zeros of polynomials sets (see e.g. Becker and Wörmann [5] and the recent work of Rouillier [11]).

The goal of this paper is to show that a triangular polynomial set $\mathbb{G}$ as in (1.1) also has several advantages from a *numerical point of view.* Indeed, it also permits us to define *multivariate Newton sums*, the multivariate analogue of Newton sums for univariate polynomials (which can be used for counting real zeros as in Gantmacher [8, Chap. 15, p. 200]). We shall see that indeed the same is true for multivariate polynomials systems in triangular form (1.1). Namely, we show that:

(a) With a triangular system $\mathbb{G}$ as in (1.1) we may associate real *moment* matrices $M_p(y)$ depending on the (known) multivariate Newton sums of $\mathbb{G}$ (to be defined later) and on an unknown vector y. The condition $M_p(y) \succeq 0$ for some specific $p = r_0$ (meaning $M_p(y)$ positive semidefinite) defines a unique solution y^*, the vector of all moments (up to order $2p$) of a probability measure μ^* supported on *all* the zeros of $\mathbb{G}$ in $\mathbb{C}^n$. As a consequence, a polynomial of degree less than $2p$ is in $\sqrt{I}$ if and only if its vector of coefficients f satisfies the linear system of equations $M_p(y^*)f = 0$.

(b) Moreover, given a set

$$\mathbb{K} := \{z \in \mathbb{C}^n \,|\, w_j(z_1, \dots, z_n, \overline{z_1}, \dots, \overline{z_n}) \geq 0,\ j = 1, \dots, m\} \subset \mathbb{C}^n,$$

defined by some polynomials $\{w_j\}$ in $\mathbb{C}[z, \overline{z}]$ (which can be viewed as a semi-algebraic set in $\mathbb{R}^{2n}$), one may also check whether the zero set of $\mathbb{G}$ is contained in $\mathbb{K}$, by solving a convex *semidefinite program* for which efficient software packages are now available. The necessary and sufficient conditions state that the system of LMI (Linear Matrix Inequalities)

$$M_{r_0}(y) \succeq 0, \quad M_{r_0}(w_i y) \succeq 0, \quad i = 1, \dots, m,$$

for some appropriate *moment matrix* $M_{r_0}(y)$ and *localizing matrices* $M_{r_0}(w_i y)$ (depending on the Newton sums of $\mathbb{G}$) must have a solution, which is then unique, i.e. $y = y^*$ with y^* as in (a). In fact, it suffices to solve the single inequality $M_p(y) \succeq 0$, which yields the unique solution y^*, and then check *afterwards* whether $M_{r_0}(w_i y^*) \succeq 0$, for all $i = 1, \dots, m$. For an introduction to semidefinite programming, the interested reader is referred to Vandenberghe and Boyd [13].

(c) As a consequence, we also provide a necessary and sufficient condition (only in terms of the Newton sums of $\mathbb{G}$) for *all* the zeros of $\mathbb{G}$ to be *real*, and also for these real zeros to be in a given semi-algebraic set

$$\mathbb{K}_1 := \{x \in \mathbb{R}^n \,|\, u_i(x_1, \dots, x_n) \geq 0, \quad i = 1, \dots, m\} \subset \mathbb{R}^n,$$

for some polynomials $\{u_i\}$ in $\mathbb{R}[x_1, \dots, x_n]$. In this case, the moment matrix is completly known and depends only on the Newton sums of $\mathbb{G}$. This latter result extends to the multivariate case a previous result of the same vein by the author for the univariate case [9].

(d) Finally, it is shown that the number of (distinct) real zeros of $\mathbb{G}$ is the maximal rank of a positive semidefinite moment matrix $M_{r_0}(y)$, that is, a y that maximizes this rank is the vector of moments of a probability measure with support on *all* the real zeros of $\mathbb{G}$. This also provides a charaterization of the ideal $I(V_{\mathbb{R}}(I))$ in terms of moment matrices.

The basic technique that we use relies on a deep result of Curto and Fialkow [6] for the $\mathbb{K}$-moment problem.

2. Notation, definitions and preliminary results

Some of the material in this section is from Curto and Fialkow [6]. Let $\mathcal{P}_r$ be the space of polynomials in $\mathbb{C}[z_1,\dots,z_n,\overline{z}_1,\dots,\overline{z}_n]$ (for short $\mathbb{C}[z,\overline{z}]$) of degree at most $r \in \mathbb{N}$. Now, following notation as in Curto and Fialkow [6], a polynomial $\theta \in \mathbb{C}[z,\overline{z}]$ is written

$$\theta(z,\overline{z}) \,=\, \sum_{\alpha\beta} \theta_{\alpha\beta}\overline{z}^{\alpha} z^{\beta} \,=\, \sum_{\alpha,\beta} \theta_{\alpha\beta}\overline{z}_1^{\alpha_1}\cdots\overline{z}_n^{\alpha_n} z_1^{\beta_1}\cdots z_n^{\beta_n},$$

in the usual basis of monomials (e.g. ordered lexicographically)

$$1, z_1, \dots, z_n, \overline{z}_1, \dots, \overline{z_n}, z_1^2, z_1 z_2, \dots . \tag{2.1}$$

We here identify $\theta \in \mathbb{C}[z,\overline{z}]$ with its vector of coefficients $\theta := \{\theta_{\alpha\beta}\}$ in the basis (2.1).

Given an infinite sequence $\{y_{\alpha\beta}\}$ indexed in the basis (2.1), we also define the linear functional on $\mathbb{C}[z,\overline{z}]$

$$\theta \,\mapsto\, \Lambda(\theta) \,:=\, \sum_{\alpha,\beta} \theta_{\alpha\beta} y_{\alpha\beta} \,=\, \sum_{\alpha,\beta} \theta_{\alpha\beta} y_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}.$$

2.1. **The moment matrix.** Given $p \in \mathbb{N}$ and an infinite sequence $\{y_{\alpha\beta}\}$, let $M_p(y)$ be the unique square matrix such that

$$\langle M_p(y) f, h\rangle \,=\, \Lambda(f\overline{h}) \quad \forall f,h \in \mathcal{P}_p$$

(see e.g. Curto and Fialkow [6, p. 3]).

To fix ideas, in the two-dimensional case, the moment matrix $M_1(y)$ is given by

$$M_1(y) \,=\, \begin{bmatrix} 1 & y_{0010} & y_{0001} & y_{1000} & y_{0100} \\ y_{1000} & y_{1010} & y_{1001} & y_{2000} & y_{1100} \\ y_{0100} & y_{0110} & y_{0101} & y_{1100} & y_{0200} \\ y_{0010} & y_{0020} & y_{0011} & y_{1010} & y_{0110} \\ y_{0001} & y_{0011} & y_{0002} & y_{1001} & y_{0101} \end{bmatrix}.$$

Thus, the entry of the moment matrix $M_p(y)$ corresponding to column $\overline{z}^{\alpha} z^{\beta}$ and row $\overline{z}^{\eta} z^{\gamma}$ is $y_{\alpha+\gamma,\beta+\eta}$, and if y is the moment vector of a measure μ on $\mathbb{C}^n$, then

$$\langle M_p(y) f, f\rangle \,=\, \Lambda(|f|^2) \,=\, \int |f|^2\, d\mu \,\geq\, 0 \quad \forall f \in \mathcal{P}_p, \tag{2.2}$$

which shows that $M_p(y)$ is *positive semidefinite* (denoted $M_p(y) \succeq 0$).

2.2. **Localizing matrices.** Let $\{y_{\alpha\beta}\}$ be an infinite sequence and let $\theta \in \mathbb{C}[z,\overline{z}]$. Define the *localizing* matrix $M_p(\theta y)$ to be the unique square matrix such that

$$\langle M_p(\theta y) f, g\rangle \,=\, \Lambda(\theta f \overline{g}) \quad \forall f,g \in \mathcal{P}_p. \tag{2.3}$$

Thus, if $\theta(z,\overline{z}) = \sum_{\alpha\beta} \theta_{\alpha\beta}\overline{z}^{\alpha} z^{\beta}$ and $M_p(y)(i,j) = y_{\gamma\eta}$, then

$$M_p(\theta y)(i,j) \,=\, \sum_{\alpha\beta} \theta_{\alpha\beta} y_{\alpha+\gamma,\beta+\eta}. \tag{2.4}$$

For instance, with $z \mapsto \theta(z,\overline{z}) := 1 - \overline{z}_1 z_1$, $M_1(\theta y)$ reads

$$\begin{bmatrix} 1 - y_{1010} & y_{0010} - y_{1020} & y_{0001} - y_{1011} & y_{1000} - y_{2010} & y_{0100} - y_{1110} \\ y_{1000} - y_{2010} & y_{1010} - y_{2020} & y_{1001} - y_{2011} & y_{2000} - y_{3021} & y_{1100} - y_{2110} \\ y_{0100} - y_{1110} & y_{0110} - y_{2020} & y_{0101} - y_{1111} & y_{1100} - y_{2110} & y_{0200} - y_{1210} \\ y_{0010} - y_{1020} & y_{0020} - y_{1030} & y_{0011} - y_{1021} & y_{1010} - y_{2020} & y_{0110} - y_{1120} \\ y_{0001} - y_{1011} & y_{0011} - y_{1021} & y_{0002} - y_{1012} & y_{1001} - y_{2011} & y_{0101} - y_{1111} \end{bmatrix}.$$

It follows that if y is the moment vector of some measure μ on $\mathbb{C}^n$, supported on the set $\{z \in \mathbb{C}^n \mid \theta(z,\overline{z}) \geq 0\}$, we then have

$$\langle M_p(\theta y) f, f \rangle \;=\; \Lambda(\theta |f|^2) \;=\; \int \theta |f|^2 \, d\mu \;\geq\; 0 \quad \forall f \in \mathcal{P}_p, \tag{2.5}$$

so that $M_p(\theta y) \succeq 0$.

2.3. **Multivariate Newton sums.** With $x_1 < x_2 < \cdots < x_n$ and given a fixed term ordering of monomials, consider a triangular polynomial system $\mathbb{G} = \{g_1, \ldots, g_n\}$ as in (1.1), that is,

$$g_i(x) \;=\; p_i(x_1, \ldots, x_{i-1})\, x_i^{r_i} + h_i(x_1, \ldots, x_i) \;=\; 0 \quad \forall i = 1, \ldots, n \tag{2.6}$$

(with $p_1 \in \mathbb{R}$), and the p_i's are such that for all $i = 2, 3, \ldots, n$,

$$g_k(z) \;=\; 0 \;\forall k = 1, \ldots, i-1 \Rightarrow p_i(z) \;\neq\; 0. \tag{2.7}$$

For each $i = 1, \ldots, n$, $p_i(x_1, \ldots, x_{i-1}) x_i^{r_i}$ is the leading term of g_i. In the terminology used in e.g. Wang [7, Definitions 2.1], $\mathbb{G}$ is a *triangular set*.

In view of the assumption on the g_i's, it follows that $\mathbb{G}$ has exactly $s := \prod_{i=1}^n r_i$ zeros $\{z(i)\}_{i=1}^s$ in $\mathbb{C}^n$ (counting their multiplicity) so that $I = \langle g_1, \ldots, g_n \rangle$ is a zero-dimensional ideal and the affine variety $V_{\mathbb{C}}(I) \subset \mathbb{C}^n$ is a finite set of cardinality $s_{\mathbb{G}} \leq s$.

For every $\alpha \in \mathbb{N}^n$ define s_α to be the real number

$$s_\alpha \;:=\; s^{-1} \sum_{i=1}^{s} z(i)^\alpha \;=\; \sum_{i=1}^{s} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}(i) \tag{2.8}$$

which we call the (normalized) α-Newton sum of $\mathbb{G}$ by analogy with the Newton sums of a univariate polynomial (see e.g. Gantmacher [8, p. 199]).

Remark 2.1. Note that the Newton sums s_α *depend* on $\mathbb{G}$ and not only on the zeros $\{z(i)\}$ because we take into account the possible multiplicities.

Proposition 2.2. *Let the g_i's be as in* (2.6)-(2.7) *and let s_α be as in* (2.8). *Then each s_α is a rational fraction in the coefficients of the g_i's and can be computed recursively.*

For a proof see §5.1.

Example 2.3. Consider the elementary example with $\mathbb{G} := \{x_1^2 + 1, x_1 x_2^2 + x_2 + 1\}$. Then, s_{i0} is just the usual (normalized) i-Newton sum of $x_1 \mapsto x_1^2 + 1$. For instance, it follows that $s_{01} = 0$, $s_{02} = 0$. Similarly, $s_{11} = -1/2$, $s_{21} = 0$, $s_{22} = 1/2$, etc.

Interestingly, given a polynomial $t \in \mathbb{R}[x_1, \ldots, x_n]$, Rouillier [11, §3] also defines *extended Newton sums* of what he calls a *multi-ensemble* associated with a set of points of $\mathbb{C}^n$. He then uses these extended Newton sums to obtain a certain triangular representation of zero-dimensional ideals.

3. Main result

In this section we assume that we are given a polynomial set $\mathbb{G} := \{g_1, \dots, g_n\}$ in the triangular form (2.6)-(2.7).

3.1. The associated moment matrix.

The idea in this section is to build up the moment matrices (defined in §2.1) associated with a particular measure μ^* on $\mathbb{C}^n$ whose support is on *all* the zeros of the polynomial set $\mathbb{G}$. That is, let $\{z(i)\}$ be the collection of $s := \prod_{j=1}^n r_j$ zeros in $\mathbb{C}^n$ of $\mathbb{G}$ (counting their multiplicity) and let μ^* to be the probability measure on $\mathbb{C}^n$ defined by

$$\mu^* := s^{-1} \sum_{i=1}^{s} \delta_{z(i)}, \tag{3.1}$$

where δ_z stands for the Dirac measure at the point $z \in \mathbb{C}^n$.

By definition of μ^*, its moments $\int z^\alpha d\mu^*$ are just the normalized α-Newton sums (2.8). Indeed,

$$s_\alpha := \int z^\alpha \, d\mu^* = s^{-1} \sum_{i=1}^{s} z(i)^\alpha. \tag{3.2}$$

If we write

$$y^*_{\alpha\beta} := \int \overline{z}^\alpha z^\beta \, d\mu^*, \quad \alpha, \beta \in \mathbb{N}^n, \tag{3.3}$$

we have

$$s_\alpha = y^*_{\alpha 0} = y^*_{0\alpha}, \quad y^*_{\alpha\beta} = y^*_{\beta\alpha}, \quad \alpha, \beta \in \mathbb{N}^n. \tag{3.4}$$

3.2. Construction of the moment matrix of μ^*.

With μ^* as in (3.1) let $\{s_\alpha, y^*_{\alpha\beta}\}$ defined in (3.2)-(3.3) be the infinite sequence of all its moments.

We then call $M_p(\mu^*)$ the moment matrix associated with μ^*, that is, in $M_p(y)$ we replace the entries $y_{0\alpha}$ or $y_{\alpha 0}$ by s_α and the other entries $y_{\alpha\beta}$ by $y^*_{\alpha\beta}$. By Proposition 2.2, the entries s_α are known and rational fractions of the coefficients of the polynomials g_i's. They can be computed numerically or symbolically. On the other hand, moments $y^*_{\alpha\beta}$ do not have a closed form expression in terms of the coefficients of polynomials g_i's.

Therefore, we next introduce a moment matrix $M_p(\mu^*, y)$ obtained from $M_p(\mu^*)$ by replacing the (unknown) entries $y^*_{\alpha\beta}$ by variables $y_{\alpha\beta}$ and look for conditions on this matrix $M_p(\mu^*, y)$ to be exactly $M_p(\mu^*)$. For instance, in the two-dimensional case, the moment matrix $M_1(\mu^*, y)$ reads

$$M_1(\mu^*, y) = \begin{bmatrix} 1 & s_{10} & s_{01} & s_{10} & s_{01} \\ s_{10} & y_{1010} & y_{1001} & s_{20} & s_{11} \\ s_{01} & y_{0110} & y_{0101} & s_{11} & s_{02} \\ s_{10} & s_{20} & s_{11} & y_{1010} & y_{0110} \\ s_{01} & s_{11} & s_{02} & y_{1001} & y_{0101} \end{bmatrix}$$

(with $s_\alpha = y_{\alpha 0} = y_{0\alpha}$). Moreover, from the definition of μ^*, we may impose $M_p(\mu^*, y)$ to be symmetric for all $p \in \mathbb{N}$, because $y^*_{\alpha\beta} = y^*_{\beta\alpha}$ for all $\alpha, \beta \in \mathbb{N}^n$ (see (3.4)).

As $\mathbb{G}$ is a triangular polynomial system in the form (2.6)-(2.7), $I = \langle g_1, \dots, g_n \rangle$ is a zero-dimensional ideal. Therefore, let $H := \{h_1, \dots, h_m\}$ be a reduced Gröbner basis of I with respect to (in short, w.r.t.) the term ordering already defined (e.g.

the lexicographical ordering $x_1 < x_2 < \cdots < x_n$). As I is zero dimensional, for every $i = 1, \ldots, n$, we may label the first n polynomials h_j of H in such a way that $x_i^{r'_i}$ is the leading term of h_i (see e.g. Adams and Loustaunau [3, Theor. 2.2.7]).

Proposition 3.1. *Let $\mathbb{G}$ be the triangular polynomial system in* (2.6)-(2.7) *(with some term ordering), and let $H = \{h_1, \ldots, h_m\}$ be its reduced Gröbner basis (with $x_i^{r'_i}$ the leading term of h_i for all $i = 1, \ldots, n$).*

Let μ^* be the probability measure defined in (3.1). *For every $\alpha, \beta \in \mathbb{N}^n$ let*

$$y^*_{\alpha\beta} := \int \overline{z}^\alpha z^\beta \, d\mu^*. \tag{3.5}$$

Then, for every $\gamma, \eta \in \mathbb{N}^n$, $y^*_{\gamma\eta}$ is a linear combination of the $y^*_{\alpha\beta}$'s with $\alpha_i, \beta_i < r'_i$ for all $i = 1, \ldots, n$, that is,

$$y^*_{\eta\gamma} = \sum_{\alpha\beta} u_{\alpha\beta}(\eta, \gamma) y^*_{\alpha\beta}, \quad \alpha_i, \beta_i < r'_i \quad \forall i = 1, \ldots, n, \tag{3.6}$$

for some scalars $\{u_{\alpha\beta}(\eta, \gamma)\}$.

Proof. Let $H = \{h_1, \ldots, h_m\}$ be the reduced Gröbner basis of I w.r.t. the term ordering, with $x_i^{r'_i}$ the leading term of h_i for all $i = 1, \ldots, n \leq m$.

For $\eta, \gamma \in \mathbb{N}^n$, write

$$z^\eta = \sum_{i=1}^m q_i(z) h_i(z) + q_\eta(z), \quad \overline{z}^\gamma = \sum_{i=1}^m v_i(\overline{z}) h_i(\overline{z}) + v_\gamma(\overline{z}),$$

for some polynomials $\{q_\eta, q_i\}$ and $\{v_\gamma, v_i\}$ in $\mathbb{R}[x_1, \ldots, n]$, that is, z^η (resp. z^γ) are *reduced* to $q_\eta(z)$ (resp. $v_\eta(z)$) w.r.t. H. Due to the special form of H, it follows that the monomials z^α of q_η, v_γ satisfy $\alpha_i < r'_i$ for all $i = 1, \ldots, n$. Hence,

$$q_\eta(z) v_\gamma(\overline{z}) = \sum_{\alpha\beta} u_{\alpha\beta}(\eta, \gamma) \overline{z}^\alpha z^\beta, \quad \alpha_i, \beta_i < r'_i \quad \forall i = 1, \ldots, n,$$

for some scalars $\{u_{\alpha\beta}(\eta, \gamma)\}$. Therefore, from the definition of μ^*,

$$\begin{aligned} y^*_{\gamma\eta} = \int z^\eta \overline{z}^\gamma \, d\mu^* &= \int \left(\sum_{i=1}^m q_i(z) h_i(z) + q_\eta(z) \right) \left(\sum_{i=1}^m v_i(\overline{z}) h_i(\overline{z}) + v_\gamma(\overline{z}) \right) d\mu^* \\ &= \int q_\eta(z) v_\gamma(\overline{z}) \, d\mu^* = \sum_{\alpha\beta} u_{\alpha\beta}(\eta, \gamma) \int \overline{z}^\alpha z^\beta \, d\mu^* \\ &= \sum_{\alpha\beta} u_{\alpha\beta}(\eta, \gamma) y^*_{\alpha\beta}, \qquad \alpha_i, \beta_i < r'_i \quad \forall i = 1, \ldots, n. \end{aligned}$$

□

The $y^*_{\alpha\beta}$'s with $\alpha_i, \beta_i < r'_i$, correspond to the *irreducible* monomials x^α, x^β with respect to the Gröbner basis H, which form a basis of $\mathbb{R}[x_1, x_2, \ldots, x_n]/I$ viewed as a vector space over $\mathbb{R}$. In fact, in view of the triangular form (2.6)-(2.7), the Gröbner basis H of I w.r.t. to the lexicographical ordering $x_1 < \cdots < x_n$ is such that $r'_i = r_i$ for all $i = 1, \ldots, n$ and H has exactly n terms (Rouillier [12]).

In view of Proposition 3.1, we may redefine the moment matrix $M_p(\mu^*, y)$ in an equivalent form as follows.

Definition 3.2 (Construction of $M_p(\mu^*, y)$)**.** Let $H = \{h_1, \ldots, h_m\}$ be a reduced Gröbner basis of I w.r.t. to the given term ordering (with $x_i^{r'_i}$ the leading term of h_i for all $i = 1, \ldots, n$).

The moment matrix $M_p(\mu^*, y)$ is the moment matrix $M_p(y)$ defined in §2.1 and where:

- every entry $y_{\alpha 0}$ or $y_{0\alpha}$ of $M_p(y)$ is replaced with the (known) α-Newton sum s_α of $\mathbb{G}$.

- every entry $y_{\gamma\eta}$ in $M_p(y)$ is replaced with the linear combination (3.6) of $\{y_{\alpha\beta}\}$ with $\alpha_i, \beta_i < r'_i$ for all $i = 1, \ldots, n$.

Thus, in this equivalent formulation, only a finite number of variables $y_{\alpha\beta}$ are involved in $M_p(\mu^*, y)$, all with $\alpha_i, \beta_i < r'_i$ for all $i = 1, \ldots, n$.

Remark 3.3. The above definition of $M_p(\mu^*, y)$ *depends* on the reduced Gröbner basis H of $\mathbb{G}$, whereas the entries s_α only depend on the g_i's.

Example 3.4. Let

$$\mathbb{G} := \{x_1^3 + x_1,\ (x_1^2 + 3)x_2^3 - x_1^2 x_2^2 + (x_1^2 - x_1 - 1)x_2 - x_1 + 1\}.$$

Then,

$$H = \{x_1^3 + x_1;\ 6x_2^3 - 3x_1^2 x_2^2 + 4x_2 x_1^2 - 3x_2 x_1 - 2x_2 - x_1^2 - 3x_1 + 2\},$$

and, for instance, denoting "$\rightsquigarrow$" as the reduction process w.r.t. H,

$$z_2^3 \rightsquigarrow (3z_1^2 z_2^2 - 4z_2 z_1^2 + 3z_2 z_1 + 2z_2 + z_1^2 + 3z_1 - 2)/6,$$

and as $z_1^4 \rightsquigarrow -z_1^2$, we have

$$y_{4003} = (-3y_{2022} + 4y_{2021} - 3y_{2011} - 2y_{2001} - y_{2020} - 3y_{2010} + 2y_{2000})/6,$$

and the latter expression can be substituted for every occurrence of y_{4003}.

Theorem 3.5. *Let $\mathbb{G}$ be a triangular polynomial system as in* (2.6)-(2.7) *and let $\{s_\alpha\}$ be the Newton sums of G in* (2.8)*. Let $M_p(\mu^*, y)$ be the moment matrix as in Definition* 3.2*, and let $r_0 := 2\sum_{j=1}^n (r'_j - 1)$. Then:*

(i) *For all $p \geq r_0$, $M_p(\mu^*, y) = M_p(\mu^*)$ if and only if $M_p(\mu^*, y) \succeq 0$.*

(ii) *For all $p \geq r_0$, $\operatorname{rank}(M_p(\mu^*)) = \operatorname{rank}(M_{r_0}(\mu^*))$, the number of distinct zeros in $\mathbb{C}^n$ of the polynomial system $\mathbb{G}$.*

(iii) *Let $f \in \mathbb{C}[z, \overline{z}]$ be of degree less than $2p$. All the zeros in $\mathbb{C}^n$ of the polynomial system $\mathbb{G}$ are zeros of f if and only if*

$$M_p(\mu^*) f = 0. \tag{3.7}$$

In particular, a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree less than $2p$ is in $\sqrt{I}$ if and only if (3.7) *holds.*

The proof is postponed to §5.2.

Remark 3.6. (a) Theorem 3.5(iii) has an equivalent formulation as follows. Let $f \in \mathbb{C}[z, \overline{z}]$ be of degree at most $2p$ and let $\hat{f}$ be its reduction w.r.t. H, the Gröbner basis of $\mathbb{G}$ defined in Proposition 3.1. Then the condition $M_p(\mu^*) f = 0$ is equivalent to $M_{r_0} \hat{f} = 0$.

(b) Given a reduced Gröbner basis H of I, the condition $M_{r_0}(\mu^*, y) \succeq 0$ in Theorem 3.5(i) is equivalent to the same condition for its submatrix $\widehat{M}_{r_0}(\mu^*, y)$ whose indices of rows and columns in the basis (2.1) correspond to *independent* monomials $\{z^\alpha\}$ which form a basis of $\mathbb{R}[x_1,\dots,x_n]/I$, their conjugate $\{\overline{z}^\alpha\}$ and the corresponding monomial products $\overline{z}^\alpha z^\beta$. Indeed, the positive semidefinite condition on the latter is equivalent to the positive semidefinite condition on the former.

Example 3.7. Consider the trivial example $\mathbb{G} := \{x^2+1\}$ so that $V_{\mathbb{C}}(I) = \{\pm i\}$. Then the condition $\widehat{M}_{r_0}(\mu^*, y) \succeq 0$ (see Remark 3.6(b)) reads

$$\widehat{M}_2(\mu^*, y) = \begin{bmatrix} 1 & 0 & 0 & y_{11} \\ 0 & y_{11} & -1 & 0 \\ 0 & -1 & y_{11} & 0 \\ y_{11} & 0 & 0 & 1 \end{bmatrix} \succeq 0,$$

which clearly implies $y_{11} = 1 = \int \overline{z}z\, d\mu^*$. Moreover,

$$\operatorname{rank}(M_{r_0}(\mu^*, y)) = \operatorname{rank}(\widehat{M}_{r_0}(\mu^*, y)) = 2 = |V_{\mathbb{C}}(I)|.$$

Similarly, let $\mathbb{G} := \{x_1^2+1, x_1x_2+1\}$ so that $V_{\mathbb{C}}(I) = \{(i,i),(-i,-i)\}$. We have $r_0 = 2$ and with the lexicographical ordering $x_1 < x_2$, $H := \{x_1^2+1, x_2 - x_1\}$ is a reduced Gröbner basis of I. Hence, in the moment matrix $M_{r_0}(\mu^*, y)$ every $y_{\alpha_1\alpha_2\beta_1\beta_2}$ is replaced with $y_{\alpha_1+\beta_1,0,\alpha_2+\beta_2,0}$. Moreover, we only need to consider $\alpha_1, \beta_1 \le 1$. Therefore, we only need to consider the monomials $\{z_1, \overline{z_1}, z_1\overline{z_1}\}$, and in view of Remark 3.6(b), the (equivalent) condition $\widehat{M}_{r_0}(\mu^*, y) \succeq 0$ reads (denoting $y_{1010} = y$)

$$\begin{bmatrix} 1 & 0 & 0 & y \\ 0 & y & -1 & 0 \\ 0 & -1 & y & 0 \\ y & 0 & 0 & 1 \end{bmatrix} \succeq 0,$$

which implies $y = 1 = \int \overline{z}_1 z_1\, d\mu^*$. Moreover,

$$\operatorname{rank}(M_{r_0}(\mu^*, y)) = \operatorname{rank}(\widehat{M}_{r_0}(\mu^*, y)) = 2 = |V_{\mathbb{C}}(I)|.$$

3.3. **Conditions for a localization of zeros of $\mathbb{G}$.** Let $w_i \in \mathbb{C}[z, \overline{z}]$, $i = 1,\dots,m$, be given polynomials and let $\mathbb{K} \subset \mathbb{C}^n$ be the set defined by

$$\mathbb{K} := \{z \in \mathbb{C}^n \mid w_i(z, \overline{z}) \ge 0, \quad i = 1,\dots,m\}. \tag{3.8}$$

We now consider the following issue:

Under what conditions on the coefficients of the polynomials g_i's are all the zeros of the triangular system $\mathbb{G}$ contained in $\mathbb{K}$?

Let $M_p(w_i y)$ be the localizing matrices (cf. §2.2) associated with the polynomials w_i, for all $i = 1,\dots,m$. As we did for the moment matrix $M_p(\mu^*, y)$ in Definition 3.2, we define $M_p(\mu^*, w_i, y)$ to be the localizing matrix $M_p(w_i y)$ where the entries $y_{0\alpha}$ and $y_{\alpha 0}$ are replaced with the α-Newton sums s_α, and where all the $y_{\eta\gamma}$ are replaced by the linear combinations (3.6) of the $\{y_{\alpha\beta}\}$ with $\alpha_i, \beta_i < r_i'$ for all $i = 1,\dots,n$. Accordingly, $M_p(\mu^*, w_i)$ is obtained from $M_p(w_i y)$ by replacing y with y^* as in Proposition 3.1.

Theorem 3.8. *Let $\mathbb{G}$ be the triangular system in* (2.6)-(2.7) *and let $M_{r_0}(\mu^*, y)$ be as in Theorem* 3.5*. Then, all the zeros of $\mathbb{G}$ are in $\mathbb{K}$ if and only if*

$$M_{r_0}(\mu^*, w_i) \succeq 0, \qquad i = 1,\dots,m. \tag{3.9}$$

Equivalently, all the zeros of $\mathbb{G}$ *are in* $\mathbb{K}$ *if and only if the system of linear matrix inequalities*

$$M_{r_0}(\mu^*, y) \succeq 0, \quad M_{r_0}(\mu^*, w_i, y) \succeq 0, \quad i = 1, \ldots, m, \tag{3.10}$$

has a solution y.

Proof. The necessity is obvious. Indeed, assume that all the zeros of $\mathbb{G}$ are in $\mathbb{K}$. Let μ^* be as in (3.1) and let $y^* := \{s_\alpha, y^*_{\alpha\beta}\}$ be the infinite sequence of moments of μ^*. Then, of course, $M_p(\mu^*) \succeq 0$ and

$$M_p(\mu^*, w_i) \,=\, M_p(w_i y^*) \succeq 0, \quad i = 1, \ldots, m,$$

for all $p \in \mathbb{N}$, is a necessary condition for μ^* to have its support in $\mathbb{K}$. Thus, $y := \{y^*_{\alpha\beta}\}$ is a solution of (3.10).

Conversely, let y be a solution of (3.10). From Theorem 3.5(i) $\{s_\alpha, y_{\alpha\beta}\}$ is the moment vector of μ^*, that is, $\{y_{\alpha\beta}\} = \{y^*_{\alpha\beta}\}$ for all α, β with $\alpha_i, \beta_i < r'_i$, for all $i = 1, \ldots, n$. Then, all the other $y^*_{\alpha\beta}$ can be obtained from the former by (3.6). Therefore, and in view of the construction of the localizing matrices $M_p(\mu^*, w_i, y)$, we have

$$M_p(\mu^*, w_i, y) \,=\, M_p(\mu^*, w_i, y^*) \,=\, M_p(\mu^*, w_i).$$

Moreover, using the terminology of Curto and Fialkow [6] (see also the proof of Theorem 3.5), all the moment matrices $M_p(\mu^*, y) = M_p(\mu^*)$ $(p > r_0)$ are *flat positive extensions* of $M_{r_0}(\mu^*, y) = M_{r_0}(\mu^*)$. As $M_{r_0}(\mu^*, w_i, y) = M_{r_0}(\mu^*, w_i) \succeq 0$, it follows from Theorem 1.6 in Curto and Fialkow [6] that μ^* has its support contained in $\mathbb{K}$. Hence, as μ^* is supported on *all* the zeros of $\mathbb{G}$, all the zeros of $\mathbb{G}$ are in $\mathbb{K}$. $\square$

3.4. **Triangular systems with only real zeros.** In this section we are interested in conditions on the coefficients of the polynomials g_i's for the triangular system $\mathbb{G}$ to have *all* its zeros *real* (i.e. in $\mathbb{R}^n$). One way to proceed is to apply Theorem 3.8 with the set $\mathbb{K}$ defined by $\mathbb{K} := \{z \in \mathbb{C}^n \,|\, w_i(z, \overline{z}) = 0,\ i = 1, \ldots, n\}$ with $z \mapsto w_i(z, \overline{z}) := z_i - \overline{z}_i$ for all $i = 1, \ldots, n$.

In this case, everything simplifies because the localizing conditions

$$M_p(\mu^*, w_i, y) \,=\, 0, \quad i \,=\, 1, \ldots, n, \quad p \in \mathbb{N}$$

(necessary for μ^* to have its support on $\mathbb{K}$), simply mean that for every $\alpha, \beta \in \mathbb{N}^n$,

$$y_{\alpha\beta} \,=\, y_{\alpha+\beta,0} \,=\, y_{0,\alpha+\beta} \,=\, s_{\alpha+\beta}.$$

In other words, we only need to deal with the Newton sums $\{s_\alpha\}$ of $\mathbb{G}$. In particular, to define $M_p(\mu^*, y)$, we do not have to introduce the reduced Gröbner basis H of I in Definition 3.2. Thus, the moment matrix $M_p(\mu^*)$ simplifies, and we only need consider the basis of monomials

$$1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_1^r, \ldots, x_n^r, \ldots \tag{3.11}$$

(without their conjugates) for the real-valued polynomials in $\mathbb{R}[x_1, \ldots, x_n]$.

Therefore, with μ^* as in (3.1), the moment matrix $M_p(\mu^*)$ is now indexed in the basis (3.11) and is completely known. Indeed,

- $M_p(\mu^*)(1, j) = s_\alpha$ if the column j corresponds to the monomial x^α in the basis (3.11), and
- if $M_p(\mu^*)(1, j) = s_\alpha$ and $M_p(\mu^*)(i, 1) = s_\beta$, then $M_p(\mu^*)(i, j) = s_{\alpha+\beta}$.

In fact, as $M_p(\mu^*)$ is completely determined from the Newton sums $\{s_\alpha\}$ of $\mathbb{G}$, let us call $M_p(s)$ the moment matrices $M_p(\mu^*)$ for all $p \in \mathbb{N}$.

Next, let $\mathbb{K}_1 \subset \mathbb{R}^n$ be the semi-algebraic set defined by

$$\mathbb{K}_1 := \{x \in \mathbb{R}^n \mid u_i(x) \geq 0, \quad i = 1, \ldots, m\},$$

for some given polynomials $u_i \in \mathbb{R}[x_1, \ldots, x_n]$, $i = 1, \ldots, m$.

We also denote by $M_p(u_i, s)$ the localizing matrix $M_p(u_i y)$ indexed in the basis (3.11), and where all the entries $\{y_\alpha\}$ are replaced with the corresponding Newton sums $\{s_\alpha\}$. We obtain

Theorem 3.9. *Let $\mathbb{G}$ be the triangular system defined in* (2.6)-(2.7) *and let $\{s_\alpha\}$ be the Newton sums of $\mathbb{G}$ defined in* (2.2). *Let $r_0 := \sum_{i=1}^n (r'_j - 1)$ with r'_j as in Theorem* 3.5.

(i) *All the zeros of $\mathbb{G}$ are real if and only if*

$$M_{r_0}(s) \succeq 0. \tag{3.12}$$

Moreover, the number of distinct zeros is $\operatorname{rank}(M_{r_0}(s))$.

(ii) *All the zeros of $\mathbb{G}$ are real and in $\mathbb{K}_1$ if and only if*

$$M_{r_0}(s) \succeq 0, \quad M_{r_0}(u_i, s) \succeq 0, \quad i = 1, \ldots, m. \tag{3.13}$$

Proof. This is just a particular case of Theorem 3.8 where the simplification is due to the localizing constraints $M_p(w_i y) = 0$ for all $i = 1, \ldots, n$, which permits us to deal only with the Newton sums $\{s_\alpha\}$ of $\mathbb{G}$. Again, as in the proof of Theworem 3.5, one uses Theorem 1.6 of Curto and Fialkow [6], but this time for measures on $\mathbb{R}^n$ and not on $\mathbb{C}^n$. □

Example 3.10. Let $\mathbb{G} := \{x_1^2 - 1, x_1 x_2^2 - 1\}$ so that $V_\mathbb{C}(I) \neq V_\mathbb{R}(I)$. In the matrix $M_{r_0}(s)$ we only need to consider its submatrix $\widehat{M_2}(s)$ with rows and columns indexed by the monomials $\{1, x_1, x_2, x_1 x_2\}$ because x_1^2 and x_2^2 are linear combinations of those monomials (see Remark 3.6(b)). Therefore,

$$\widehat{M_2}(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and obviously, $M_2(s) \succeq 0$ does not hold.

Now with $\mathbb{G} := \{x_1^2 - 1, x_1 x_2 - 1\}$ we have $V_\mathbb{C}(I) = V_\mathbb{R}(I) = \{(1,1), (-1,-1)\}$, and we obtain

$$\widehat{M_1}(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \succeq 0,$$

with $\operatorname{rank}(\widehat{M_1}(s)) = 2 = |V_\mathbb{R}(I)|$.

Theorem 3.9 is the analogue in the multivariate case of the result in Lasserre [9] in which one obtains a similar necessary and sufficient condition on the Newton sums of a univariate polynomial g, for g to have all its zeros real and in a prescribed interval $[a, b]$. In the univariate case, and with $\mathbb{G} = \{g\}$ for a single univariate polynomial g of degree $n + 1$, one may check that $(n + 1)M_n(s)$ is just the (Hankel) matrix associated with Hermite's quadratic form $\operatorname{Her}(g, 1)$ (see [4, p. 99]). Similarly, given another univariate polynomial h, $(n + 1)M_n(h, s)$ is the matrix associated with Hermite's quadratic form $\operatorname{Her}(g, h)$ and whose *signature* gives the number of *real* zeros of g that satisfy $h(x) > 0$ minus the number of real zeros that satisfy $h(x) < 0$

([4, Theorem 4.13]). Both $M_n(s)$ and $M_n(h,s)$ are explicit in terms of standard Newton sums.

In the multivariate case, let $\widehat{M}_{r_0}(s)$ (resp. $\widehat{M}_{r_0}(u_i,s)$) be the submatrix obtained from $M_{r_0}(s)$ (resp. $M_{r_0}(u_i,s)$) by keeping only the rows and columns indexed by monomials $\{x^\alpha\}$ which form a basis of $\mathbb{R}[x_1,\dots,x_n]/I$ as an $\mathbb{R}$-vector space. Then, one may check that (after scaling) $\widehat{M}_{r_0}(s)$ is the matrix associated with the multivariate Hermite's quadratic form $\mathrm{Her}(\mathbb{G},1)$ (see [4, p. 129]). Similarly (after scaling again), $\widehat{M}_{r_0}(u_i,s)$ is the matrix associated with the multivariate Hermite's quadratic form $\mathrm{Her}(\mathbb{G},u_i)$, and whose *signature* gives the number of *real* zeros of $\mathbb{G}$ that satisfy $u_i(x)>0$ minus the number of real zeros that satisfy $u_i(x)<0$ ([4, Theorem 4.72]). Here, and as in the univariate case, both $\widehat{M}_{r_0}(s)$ and $\widehat{M}_{r_0}(u_i,s)$ are obtained *explicitly* in terms of generalized Newton sums, because of the *triangular* form of $\mathbb{G}$. Note that in Theorem 3.9, we do *not* need to determine a basis of $\mathbb{R}[x_1,\dots,x_n]/I$.

3.5. **Counting real zeros.** We still consider a triangular system $\mathbb{G}$ as in (2.6)-(2.7) and now consider the issue of *counting* the real zeros of $\mathbb{G}$.

As we did for μ^*, we build up the moment matrix of a probability measure μ with support on the *real* zeros of $\mathbb{G}$. This time, we cannot use the Newton sums $\{s_\alpha\}$ in (2.2) because some zeros of $\mathbb{G}$ may not be real. Therefore, we replace s_α with the unknown y_α. Namely, we define the moment matrix $M_p(y)$ as follows.

Definition 3.11. Let $H:=\{h_1,\dots,h_m\}$ be a reduced Gröbner basis of I w.r.t. some term ordering (with $x_i^{r_i'}$ the leading term of h_i for all $i=1,\dots,n\le m$). Then $M_p(y)$ is the moment matrix defined in (2.1) but now with rows and columns indexed in the basis (3.11), and where:

For every $\alpha\in\mathbb{N}^n$, the monomial y_α is replaced by a linear combination of the variables y_β with $\beta_i<r_i'$ for all $i=1,\dots,n$ (see Proposition 3.1) coming from the reduction of the monomial z^α w.r.t. H. That is, if $z^\alpha\rightsquigarrow\sum_\beta u_\beta(\alpha)z^\beta$ with $\beta_i<r_i'$ for all i, and for some scalars $\{u_\beta(\alpha)\}$, then y_α is replaced with $\sum_\beta u_\beta(\alpha)y_\beta$.

We denote by $V_{\mathbb{R}}(I)\subset\mathbb{R}^n$ the set of real zeros of $\mathbb{G}$ and $I(V_{\mathbb{R}}(I))\subset\mathbb{R}[x_1,\dots,x_n]$ the ideal generated by the variety $V_{\mathbb{R}}(I)$. Recall that a polynomial $f\in\mathbb{R}[x_1,\dots,x_n]$ is identified with its vector (also denoted by f) of coefficients in the basis (3.11).

Proposition 3.12. *Let $\mathbb{G}$ be a triangular system as in* (2.6)-(2.7) *and let $M_p(y)$ be as in Definition* 3.11*, and $r_0:=\sum_{i=1}^n(r_i'-1)$. Then:*

(a) *The number s_0 of distinct real zeros of $\mathbb{G}$ is given by the maximal rank of $M_{r_0}(y)$ over all possible solutions y (if any) of $M_{r_0}(y)\succeq 0$.*

(b) *Let y be such that $M_{r_0}(y)\succeq 0$ with $\mathrm{rank}(M_{r_0}(y))=s_0$. Let $f\in\mathbb{R}[x_1,\dots,x_n]$ be a polynomial of degree less than r_0. Then,*

$$f\in I(V_{\mathbb{R}}(I))\quad\Leftrightarrow\quad M_{r_0}(y)f\;=\;0. \tag{3.14}$$

Proof. (a) Assume that there is a solution y to $M_{r_0}(y)\succeq 0$. Then, proceding as in the proof of Theorem 3.5, using Theorem 1.6 in Curto and Fialkow [6], it follows that y is the vector of moments of a $\mathrm{rank}(M_{r_0}(y))$-atomic probability measure μ, *this time on* $\mathbb{R}^n$, and with support on the real zeros of $\mathbb{G}$. Therefore, the rank of $M_{r_0}(y)$ is not larger than the number s_0 of distinct real zeros of $\mathbb{G}$.

Next, let $\{x(i)\}_{i=1}^{s_0}$ be the real distinct zeros of $\mathbb{G}$ and let $\mu:=s_0^{-1}\sum_{i=1}^{s_0}\delta_{x(i)}$ (with δ_x the Dirac measure at the point $x\in\mathbb{R}^n$). Let y be the infinite sequence of all the moments of μ. It follows easily that the moment matrices $M_p(y)$ are exactly as in Definition 3.11, and moreover, $M_{r_0}(y) \succeq 0$ holds, as it is a necessary condition for y to be a moment sequence. As (from its definition) μ is an s_0-atomic probability measure with support on the distinct real zeros of $\mathbb{G}$, we conclude from what precedes that $\mathrm{rank}(M_{r_0}(y)) = s_0$.

(b) Let $f \in I(V_{\mathbb{R}}(I))$ and let y be as in Proposition 3.12(b). From (a) it follows that y is the sequence of moments (up to order $2r_0$) of a probability measure μ_y with support on the s_0 distinct real zeros of $\mathbb{G}$ (that is, with support on *all* the points of $V_{\mathbb{R}}(I)$). Hence, from (2.2)

$$0 = \int f^2 \, d\mu_y = \langle M_{r_0}(y)f, f\rangle \Rightarrow M_{r_0}(y)f = 0 \quad \text{because } M_{r_0}(y) \succeq 0.$$

Conversely, let f be such that $M_{r_0}(y)f = 0$. Then

$$\langle M_{r_0}(y)f, f\rangle = 0 \Rightarrow \int f^2 \, d\mu_y = 0 \Rightarrow f \equiv 0 \quad \mu_y\text{-almost everywhere,}$$

which implies that $f(x) = 0$ for all $x \in V_{\mathbb{R}}(I)$, or, equivalently, $f \in I(V_{\mathbb{R}}(I))$. $\square$

Remark 3.13. (i) The assumption on the degree of f in Proposition 3.12(b) is not restrictive, for one may first reduce f w.r.t. the Gröbner basis H of I to end up with a polynomial of degree less than r_0.

(ii) Proposition 3.12(a) should not be misleading. Finding a vector y such that $M_{r_0}(y) \succeq 0$ has *maximal rank* is not necessarily easy. (However, note that SDP solvers that use interior point methods usually find solutions with highest rank.) Proposition 3.12(a)-(b) should be viewed as an alternative characterization of the number of real zeros of $\mathbb{G}$ and of the ideal $I(V_{\mathbb{R}}(I))$ in terms of *moment matrices*. Note that in contrast to counting techniques via multivariate Hermite's quadratic form, knowledge of a basis of $\mathbb{R}[x_1, \ldots, x_n]/I$ is not needed in Proposition 3.12.

4. Conclusion

In this paper we have considered a system $\mathbb{G}$ of polynomial equations in triangular form and show that several characterizations of the zeros of $\mathbb{G}$ may be obtained from *positive semidefinite* (numerical) conditions on appropriate *moment* and *localizing* matrices. In particular, the triangular form of $\mathbb{G}$ permits us to define the analogue for the multivariate case of *Newton sums* of a univariate polynomial. As in the univariate case, these multivariate Newton sums permit us to give explicit conditions on the coefficients of the polynomials g_i's for $\mathbb{G}$ to have only real zeros, and for those zeros to be in a given semi-algebraic set of $\mathbb{R}^n$.

5. Proofs

5.1. **Proof of Proposition 2.2.**

Proof. The proof is by induction. In view of the triangular form (2.6)-(2.7), the zero set of $\mathbb{G}$ in $\mathbb{C}^n$ (or, equivalently, the variety $V_{\mathbb{C}}(I)$ associated with I) consists of $s := \prod_{j=1}^n r_j$ zeros that we label $z(i)$, $i = 1, \ldots, s$, counting their multiplicity.

In addition, still in view of (2.6)-(2.7), any particular zero $z(i) \in \mathbb{C}^n$ of $\mathbb{G}$ can be written

$$z(i) = [z_1(i_1), z_2(i_1, i_2), \ldots, z_n(i_1, \ldots, i_n)],$$

for some multi-index $i_1 \leq r_1, \ldots, i_n \leq r_n$, and where each $z_k(i_1, \ldots, i_k) \in \mathbb{C}$ is a zero of the univariate polynomial $x \mapsto g_k(z_1(i_1), \ldots, z_{k-1}(i_1, \ldots, i_{k-1}), x)$ (where multiplicy is taken into account).

Therefore, for every $\alpha \in \mathbb{N}^n$, the α-Newton sum y_α defined in (2.8) can be written

$$sy_\alpha := \sum_{i=1}^{s} z(i)^\alpha = \sum_{i_1 \leq r_1, \ldots, i_n \leq r_n} z_1(i_1)^{\alpha_1} z_2(i_1, i_2)^{\alpha_2} \cdots z_n(i_1, \ldots, i_n)^{\alpha_n}. \tag{5.1}$$

Let us make the following induction hypothesis.

H_k. *For every $p, q \in \mathbb{R}[x_1, \ldots, x_k]$*

$$\begin{aligned} S_k(p,q) &:= \sum_{i_1,\ldots,i_k} \frac{p(z_1(i_1), \ldots, z_k(i_k))}{q(z_1(i_1), \ldots, z_k(i_k))} \\ &= \sum_{i_1,\ldots,i_k} \frac{\sum_\alpha p_\alpha z_1(i_1)^{\alpha_1} \cdots z_k(i_k)^{\alpha_k}}{\sum_\alpha q_\alpha z_1(i_1)^{\alpha_1} \cdots z_k(i_k)^{\alpha_k}} \end{aligned} \tag{5.2}$$

is a rational fraction of coefficients of the polynomials g_i's, $i = 1, \ldots, k$.

Observe that (5.1) is a particular case of (5.2) in H_n.

We first prove that H_1 is true. Let $p, q \in \mathbb{R}[x_1]$ and

$$S(p,q) = \sum_{j=1}^{r_1} \frac{\sum_k p_k z_1(j)^k}{\sum_k q_k z_1(j)^k},$$

with $\{z_1(j)\}$ being the zeros of $x_1 \mapsto g_1(x_1)$, counting their multiplicity.

Reducing to a common denominator, $S(p,q)$ reads

$$S(p,q) = \frac{P(z_1(1), \ldots, z_1(r_1))}{Q(z_1(1), \ldots, z_1(r_1))},$$

for some *symmetric* polynomials P, Q of the variables $\{z_1(j)\}$ and whose coefficients are polynomials of coefficients of p, q.

Therefore, by the fundamental theorem of symmetric functions, both numerator $P(.)$ and denominator $Q(.)$ are rational fractions of coefficients of g_1 (polynomials if g_1 is monic). Thus, H_1 is true, and we can write $S_1(p,q) = u_{pq}(g_1)/v_{pq}(g_1)$ for some polynomials u_{pq}, v_{pq} of coefficients of g_1. The coefficients of u_{pq}, v_{pq} are themselves polynomials of coefficients of the polynomials p, q.

Next, assume that H_j is true for all $1 \leq j \leq k$, that is, for all $j = 1, \ldots, k$ and $p, q \in \mathbb{R}[x_1, \ldots, x_j]$,

$$S_j(p,q) = u_{pq}(g_1, \ldots, g_j)/v_{pq}(g_1, \ldots, g_j) \tag{5.3}$$

for some polynomials u_{pq}, v_{pq} of coefficients of the polynomials $g_1, \ldots, g_j$.

We are going to show that H_{k+1} is true. Let $p, q \in \mathbb{R}[x_1, \ldots, x_{k+1}]$ and let

$$S_{k+1}(p,q) = \sum_{i_1,\ldots,i_{k+1}} \frac{\sum_\alpha p_\alpha z_1(i_1)^{\alpha_1} \cdots z_{k+1}(i_1, \ldots, i_{k+1})^{\alpha_{k+1}}}{\sum_\alpha q_\alpha z_1(i_1)^{\alpha_1} \cdots z_{k+1}(i_1, \ldots, i_{k+1})^{\alpha_{k+1}}}.$$

$S_{k+1}(p,q)$ can be rewritten as

$$S_{k+1}(p,q) = \sum_{i_1,\ldots,i_k} \left[\sum_{j=1}^{r_{k+1}} \frac{\sum_\alpha p_\alpha z_1(i_1)^{\alpha_1} \cdots z_k(i_1, \ldots, i_k)^{\alpha_k} z_{k+1}(i_1, \ldots, i_k, j)^{\alpha_{k+1}}}{\sum_\alpha q_\alpha z_1(i_1)^{\alpha_1} \cdots z_k(i_1, \ldots, i_k)^{\alpha_k} z_{k+1}(i_1, \ldots, i_k, j)^{\alpha_{k+1}}} \right]. \tag{5.4}$$

In (5.4), the term

$$A := \left[\sum_{j=1}^{r_{k+1}} \frac{\sum_\alpha p_\alpha z_1(i_1)^{\alpha_1}\cdots z_k(i_1,\ldots,i_k)^{\alpha_k} z_{k+1}(i_1,\ldots,i_k,j)^{\alpha_{k+1}}}{\sum_\alpha q_\alpha z_1(i_1)^{\alpha_1}\cdots z_k(i_1,\ldots,i_k)^{\alpha_k} z_{k+1}(i_1,\ldots,i_k,j)^{\alpha_{k+1}}}\right]$$

can in turn be written as

$$A \,=\, \sum_{j=1}^{r_{k+1}} \frac{\tilde{p}(z_{k+1}(i_1,\ldots,i_k,j))}{\tilde{q}(z_{k+1}(i_1,\ldots,i_k,j))}, \tag{5.5}$$

for some univariate polynomials $\tilde{p},\tilde{q} \in \mathbb{R}[x]$ of the variable $z_{k+1}(i_1,\ldots,i_k,j)$ (which is a zero of the univariate polynomial $x \mapsto g_{k+1}(z_1(i_1),\ldots,z_k(i_1,\ldots,i_k),x)$) and whose coefficients are polynomials in the variables $z_1(i_1), z_2(i_1,i_2),\ldots,z_k(i_1,\ldots,i_k)$. In view of H_1

$$A \,=\, \frac{u_{\tilde{p}\tilde{q}}(g_{k+1})}{v_{\tilde{p}\tilde{q}}(g_{k+1})},$$

for some polynomials $u_{\tilde{p}\tilde{q}}, v_{\tilde{p}\tilde{q}}$ of the coefficients of g_{k+1}.

The coefficients of the polynomials $u_{\tilde{p}\tilde{q}}, v_{\tilde{p}\tilde{q}}$ are themselves polynomials of coefficients of p,q and of $z_1(i_1),\ldots,z_k(i_1,\ldots,i_k)$. Hence, substituting for A in (5.4) we obtain

$$\begin{aligned} S_{k+1}(p,q) \quad &= \quad \sum_{i_1,\ldots,i_k} \frac{\sum_\alpha U_\alpha(g_{k+1}) z_1(i_1)^{\alpha_1}\cdots z_k(i_1,\ldots,i_k)^{\alpha_k}}{\sum_\alpha V_\alpha(g_{k+1}) z_1(i_1)^{\alpha_1}\cdots z_k(i_1,\ldots,i_k)^{\alpha_k}} \\ &= \quad S_k(U(g_{k+1}),V(g_{k+1})) \end{aligned}$$

for some polynomials $U,V \in \mathbb{R}[x_1,\ldots,x_k]$ whose coefficients are polynomials of coefficients of g_{k+1}.

We next use the induction hypothesis H_k by which $S_k(U(g_{k+1}),V(g_{k+1}))$ is a rational fraction $f_{UV}(g_1,\ldots,g_k)/h_{UV}(g_1,\ldots,g_k)$ of coefficients of the polynomials $g_1,\ldots,g_k$. As the coefficients of f_{UV},h_{UV} are themselves rational fractions of coefficients of g_{k+1} we finally obtain that

$$S_{k+1}(p,q) \,=\, \frac{u'_{pq}(g_1,\ldots,g_{k+1})}{v'_{pq}(g_1,\ldots,g_{k+1})},$$

that is, a rational fraction of coefficients of the polynomials $g_1,\ldots,g_{k+1}$. Hence H_{k+1} is true, and therefore, the induction hypothesis is true.

Now Proposition 2.2 follows from H_n and the expression (5.1) for the α-Newton sum y_α. That the $\{y_\alpha\}$ can be computed recursively is clear from the above proof of the induction hypothesis H_k. $\square$

5.2. **Proof of Theorem 3.5.**

Proof. (i) Let $p > r_0$ be fixed, arbitrary, and write

$$M_p(\mu^*,y) \,=\, \left[\begin{array}{ccc} M_{r_0}(\mu^*,y) & | & B \\ - & & - \\ B' & | & C \end{array}\right].$$

Consider an arbitrary column $\left[\begin{array}{c} B(.,j) \\ C(.,j) \end{array}\right]$. By definition of the moment matrix, $B(1,j)$ is a monomial $z^\gamma \overline{z}^\eta$ for which $\gamma_i > r'_i$ or $\eta_k > r'_k$ for at least one index i or

k. By Proposition 3.1

$$z^\gamma \overline{z}^\eta = \sum_{\alpha,\beta} u_{\alpha\beta}(\eta,\gamma)\overline{z}^\alpha z^\beta, \quad \alpha_i, \beta_i < r_i', \ \forall i = 1,\ldots,n, \tag{5.6}$$

for some scalars $\{u_{\alpha\beta}(\eta,\gamma)\}$, so that, from the construction of $M_p(\mu^*,y)$, we have

$$\begin{aligned} B(1,j) = y_{\eta\gamma} &= \sum_{\alpha,\beta} u_{\alpha\beta}(\eta,\gamma) y_{\alpha\beta} \\ &= \sum_{\alpha,\beta} u_{\alpha\beta}(\eta,\gamma) M_{r_0}(\mu^*,y)(1,j_{\alpha\beta}), \end{aligned}$$

where $j_{\alpha\beta}$ is the index of the column of $M_{r_0}(\mu^*,y)$ corresponding to the monomial $\overline{z}^\alpha z^\beta$. Next, consider an element $B(k,j)$ of the column $B(.,j)$. The element k of $M_p(\mu^*,y)(k,1)$ is a monomial $z^p\overline{z}^q$ and from the definition of $M_p(\mu^*,y)$, we have $B(k,j) = y_{\eta+q,\gamma+p}$. Now, from (5.6) we have

$$z^p\overline{z}^q z^\gamma\overline{z}^\eta = \sum_{\alpha,\beta} u_{\alpha\beta}(\eta,\gamma) z^{\beta+p}\overline{z}^{\alpha+q},$$

which implies

$$y_{\eta+q,\gamma+p} = \sum_{\alpha,\beta} u_{\alpha\beta}(\eta,\gamma) y_{\alpha+q,\beta+p},$$

or, equivalently,

$$B(k,j) = \sum_{\alpha,\beta} u_{\alpha\beta}(\eta,\gamma) M_{r_0}(k,j_{\alpha\beta}).$$

The same argument holds for $C(.,j)$. Hence,

$$\begin{bmatrix} B \\ C \end{bmatrix}(j) = \sum_{\alpha,\beta} u_{\alpha\beta}(\eta,\gamma) \begin{bmatrix} M_{r_0}(\mu^*,y) \\ B' \end{bmatrix}(j) \quad \forall j,$$

which, in view of $M_p(\mu^*,y) \succeq 0$, implies that

$$\operatorname{rank}(M_p(\mu^*,y)) = \operatorname{rank}(M_{r_0}(\mu^*,y)).$$

As $p > r_0$ was arbitrary, and using the terminology of Curto and Fialkow [6], it follows that the matrices $M_p(\mu^*,y)$ are *flat positive extensions* of $M_{r_0}(\mu^*,y)$ for all $p > r_0$. This in turn implies that, indeed, the entries of $M_{r_0}(\mu^*,y)$ are moments of some $\operatorname{rank}(M_{r_0}(\mu^*,y))$-atomic probability measure μ.

We next prove that $\mu = \mu^*$, i.e., the condition $M_{r_0}(\mu^*,y) \succeq 0$ determines a unique vector $y = y^*$ that corresponds to the vector of moments of μ^*, up to order $2r_0$.

Given the Gröbner basis $H = \{h_i\}_{i=1}^m$ of $I = \langle g_1,\ldots,g_n\rangle$ (already considered in the proof of Proposition 3.1), let $\overline{h}_i \in \mathbb{C}[z,\overline{z}]$ be the conjugate polynomial of h_i, i.e., $\overline{h}_i(z,\overline{z}) = h_i(\overline{z})$ for all $i = 1,\ldots,m$.

We first prove that

$$M_p(h_i y) = 0, \quad M_p(\overline{h}_i y) = 0, \quad i = 1,\ldots,m, \quad p \in \mathbb{N}, \tag{5.7}$$

where $M_p(h_i y)$ (resp. $M_p(\overline{h}y)$) is the *localizing matrix* associated with the polynomials h_i (resp. $\overline{h}_i$).

By Proposition 3.1, recall that any entry $y_{\eta\gamma}$ of $M_p(\mu^*,y)$ is replaced by a linear combination of the $y_{\alpha\beta}$'s with $\alpha_i,\beta_i < r_i'$ for all $i = 1,\ldots,n$. This linear combination is coming from the *reduction* of the monomials $\{z^\alpha\}_{\alpha\in\mathbb{N}^n}$ with respect to H;

that is, let us call J the set of indices β corresponding to the irreducible monomials z^β w.r.t. H. Then, the reduction of z^α w.r.t. H yields

$$z^\alpha \,=\, \sum_{i=1}^m q_i(z)h_i(z) + \sum_{\beta\in J} u_\beta(\alpha) z^\beta \quad \text{denoted } z^\alpha \,\rightharpoonup\, \sum_{\beta\in J} u_\beta(\alpha) z^\beta,$$

and similarly,

$$\overline{z}^\alpha \,=\, \sum_{i=1}^m q_i(\overline{z})h_i(\overline{z}) + \sum_{\beta\in J} u_\beta(\alpha) \overline{z}^\beta \quad \text{denoted } \overline{z}^\alpha \,\rightharpoonup\, \sum_{\beta\in J} u_\beta(\alpha) \overline{z}^\beta.$$

From this, we obtain (see the proof of Proposition 3.1)

$$z^\gamma \overline{z}^\eta \,\rightharpoonup\, \left(\sum_{\beta\in J} u_\beta(\gamma) z^\beta\right)\left(\sum_{\beta\in J} u_\beta(\eta) \overline{z}^\beta\right) \,\rightharpoonup\, \sum_{\alpha,\beta\in J} u_{\alpha\beta}(\eta,\gamma)\overline{z}^\alpha z^\beta, \tag{5.8}$$

for some scalars $\{u_{\alpha\beta}(\eta,\gamma)\}$, and thus the entry $y_{\eta\gamma}$ of $M_p(\mu^*,y)$ is replaced with $\sum_{\alpha,\beta\in J} u_{\alpha\beta}(\eta,\gamma) y_{\alpha\beta}$, or, equivalently,

$$y_{\eta\gamma} - \sum_{\alpha,\beta\in J} u_{\alpha\beta}(\eta,\gamma) y_{\alpha\beta} \,=\, 0. \tag{5.9}$$

So let $p\in\mathbb{N}$ be fixed, and consider the entry $M_p(h_i y)(k,l)$ of the localizing matrix $M_p(h_i y)$. Recall that $M_p(y)(k,l) = y_{\phi\zeta}$ for some $\phi,\zeta\in\mathbb{N}^n$, and so $M_p(h_i y)(k,l)$ is just the expression $\overline{z}^\phi z^\zeta h_i(z)$, where each monomial $\overline{z}^\alpha z^\beta$ is replaced with $y_{\alpha\beta}$; see (2.4). Next, by definition, $\overline{z}^\phi z^\zeta h_i \rightharpoonup 0$ for all $i=1,\dots,m$. Therefore, when y is as in Definition 3.2 (that is, when (5.9) holds), writing

$$\overline{z}^\phi\, z^\zeta h_i \,=\, \sum_{\eta,\gamma\in\mathbb{N}^n} v_{\eta\gamma}\overline{z}^\eta z^\gamma \,\rightharpoonup\, 0,$$

and using (5.8)-(5.9), yields

$$M_p(h_i y)(k,l) \,=\, \sum_{\alpha,\beta\in J}\left(\sum_{\eta,\gamma\in\mathbb{N}^n} v_{\eta\gamma} u_{\alpha\beta}(\eta,\gamma)\right) y_{\alpha\beta} \,=\, 0.$$

Recall that $p\in\mathbb{N}$, and k,l were arbitrary. Therefore, when y is as in Definition 3.2, we have $M_p(h_i y) = 0$ (and similarly, $M_p(\overline{h}_i y) = 0$), for all $i=1,\dots,m$ and all $p\in\mathbb{N}$. That is, (5.7) holds.

Hence, let μ be the r-atomic probability measure encountered earlier (with $r := \operatorname{rank}(M_{r_0}(\mu^*,y))$), and let $\{z(k)\}_{k=1}^r \subset \mathbb{C}^n$ be the r distinct points of the support of μ, that is,

$$\mu = \sum_{k=1}^r u_k \delta_{z(k)}, \quad \sum_{k=1}^r u_k \,=\, 1, \qquad 0 < u_k, \quad k=1,\dots,r,$$

with $\delta_\bullet$ the Dirac measure at $\bullet$.

For every $1\le i\le r$, let $q_i \in \mathbb{C}[z,\overline{z}]$ be an interpolation polynomial that vanishes at all $z(k)$, $k\neq i$, and with $q_i(z(i),\overline{z(i)}) \neq 0$. Let $p \geq \deg q_i$. Then for all $j=1,\dots,m$, we have (also denoting q_i as the vector of coefficients of $q_i\in\mathbb{C}[z,\overline{z}]$)

$$0 \,=\, \langle q_i, M_p(h_j y) q_i\rangle \,=\, \int |q_i(z,\overline{z})|^2 h_j(z)\,\mu(dz) \,=\, u_i |q_i(z(i),\overline{z(i)})|^2 h_j(z(i)),$$

and so, $h_j(z_i) = 0$ for all $j=1,\dots,m$.

As this is true for all $1 \leq i \leq r$, it follows that

$$h_j(z(i)) = 0, \quad i = 1, \ldots, r, \quad j = 1, \ldots, m,$$

that is, μ has its support contained in $\mathbb{G}$. Therefore, with $\{z(i)\}_{i=1}^{s_0}$ being the distinct zeros in $\mathbb{C}^n$ of $\mathbb{G}$,

$$\mu = \sum_{i=1}^{s_0} u_i \delta_{z(i)}, \quad \sum_{i=1}^{s_0} u_i = 1, \quad u_i \geq 0, \quad i = 1, \ldots, n,$$

for some nonnegative scalars $\{u_i\}$, whereas $\mu^* = s^{-1} \sum_{i=1}^{s} \delta_{z(i)}$ (counting their multiplicity) or $\mu^* = \sum_{i=1}^{s_0} v_i \delta_{z(i)}$ for some nonnegative scalars $\{v_i\}$.

Remember that by definition of the matrices $M_{r_0}(\mu^*)$ and $M_{r_0}(\mu^*, y)$, their entries $\{s_\alpha\}$ (corresponding to the Newton sums) are the same. That is,

$$s_\alpha = \int z^\alpha \, d\mu = \int z^\alpha \, d\mu^*, \qquad \alpha_j \leq r_j - 1, \quad j = 1, \ldots, n.$$

Now, we also know that s_0 is less than the number of *independent* monomials $\{z^{\beta^{(j)}}\}$ (w.r.t. H) which form a basis of $\mathbb{R}[x_1, \ldots, x_n]/I$ (with equality if $I = \sqrt{I}$). Therefore, if $\mu \neq \mu^*$, we have

$$\sum_{i=1}^{s_0} (u_i - v_i) z(i)^{\beta^{(j)}} = 0, \quad j = 1, \ldots, s_0, \text{ with } u \neq v,$$

which yields that the square matrix of the above linear system is singular. Hence some linear combination $\{\lambda_j\}$ of its rows vanishes, i.e.,

$$\sum_{j=1}^{s_0} \lambda_j z(k)^{\beta^{(j)}} \quad \forall k = 1, \ldots, s_0,$$

in contradiction with the linear independence of the $\{z^{\beta^{(j)}}\}$. Hence $u = v$, which in turn implies $\mu = \mu^*$. So it follows that $M_{r_0}(\mu^*, y) \succeq 0$ has only one solution, namely $y = y^*$, the (truncated) vector y^* of moments up to order $2r_0$, of the probability measure μ^*.

Finally, this implies that $s_0 = r = \operatorname{rank}(M_{r_0}(\mu^*, y)) = \operatorname{rank}(M_{r_0}(\mu^*))$ because by Curto and Fialkow [6, Theor. 1.6], the number of atoms of $\mu = \mu^*$ is precisely $\operatorname{rank}(M_{r_0}(\mu^*, y))$. This also proves that $M_{r_0}(\mu^*, y) = M_{r_0}(\mu^*)$ and thus, (i) and (ii).

To prove (iii), consider a polynomial $f \in \mathbb{C}[z, \overline{z}]$ of degree less than $2p$ with coefficient vector in the basis (2.1) still denoted f. It is clear that if $f(z(i)) = 0$ for all $i = 1, \ldots, s_0$, then

$$0 = \int |f|^2 \, d\mu^* = \langle M_p(\mu^*) f, f \rangle,$$

which in turn implies $M_p(\mu^*) f = 0$. Conversely,

$$M_p(\mu^*) f = 0 \Rightarrow 0 = \langle M_p(\mu^*) f, f \rangle = \int |f|^2 \, d\mu^*,$$

which in turn implies $f(z) = 0$, μ^*-a.e.

Finally, let $f \in \mathbb{R}[x_1, \ldots, x_n]$. Recall that $\sqrt{I} = I(V_{\mathbb{C}}(I))$ where $V_{\mathbb{C}}(I) = \{z(i)\}_{i=1}^{s_0}$, that is, $f \in \sqrt{I}$ if and only if $f(z(i)) = 0$ for all $i = 1, \ldots, s_0$. In view of what precedes, $f \in \sqrt{I}$ if and only if $M_p(\mu^*) f = 0$. $\square$

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